PART A: Review of the Basics

Besides the linear function the following functions are MOST useful. YOUR STUDENTS MUST KNOW THEM. Can you make a quick sketch of each?

(a) $y = x^2$

(b) $y = x^3$

(c) $y = x^4$

(d) $y = \sqrt{x}$

(e) $y = \sqrt[3]{x}$

(f) $y = |x|$ (absolute value function)

(g) $y = \frac{1}{x}$

(h) $y = \frac{1}{x^2}$

A function is called an ODD FUNCTION, if $f(-x) = -f(x)$, e.g. $f(x) = x^3$, Sin x, x

A function is called an EVEN FUNCTION, if $f(-x) = f(x)$, e.g. $f(x) = x^2$, x^4, $|x|$, Cos x

Which ones of the above functions are odd, which ones are even? Odd _ _ _ _, Even _ _ .

Sketch $y = x^2$ $y = x^3$ $y = x^4$ $y = x^5$

$y = x^{12}$ $y = x^{13}$

observation: If n is odd, the graph is like _________.

If n is even , the graph is like ________.

BASIC TRANSFORMATIONS.

(a) $y = x^2$ $y = x^2 + 7$, (the graph moves ____)

$y = x^2 - 4$, (the graph moves_ ___)

(b) $y = x^2$ $y = (x-2)^2$
(replacing x by x –2. moves the graph ___________)

$y = (x+3)^2$
(replacing x by x+ 3 the graph moves _____________)

(c) From $y = x^2$ to $y = a\,x^2$ (Multiplying by a , where a is a constant).

If $a > 1$ e.g. $y = 3\,x^2$ the graph -----------.

If $0 < a < 1$ e.g. $y = \frac{3}{5}x^2$ the graph --------- -.

If $a < 0$, & $|a| > 1$, e.g. $y = -3\,x^2$, the graph -----.

If $a < 0$, & $|a| < 1$, e.g. $y = -\frac{2}{5}x^2$, the graph ------.

Practice:

$y = -3x^2$ $y = -\frac{2}{3}x^2$ $y = 5\,x^2$ $y = .35\,x^2$

Practice:

(a) $y = (x - 3)^2 + 6$

(b) $y = (x + 2)^3 - 1$

(c) $y = |x - 3| + 4$

(d) $y = -\sqrt[3]{x + 2} + 4$

PART B: Graphing POLYNOMIALS

I. Graph the following polynomials. Write your comments/observations on the right side.]

1. $y = (x-3)(x-5)$ (The graphs "behaves like" $y = x^2$. We will call this RIGHT and LEFT BEHAVIOR. It has zeros at 3, and 5)

2. $y = (x-2)(x-5)(x-6)$ (Behaves like $y = x^3$, zeros at____)

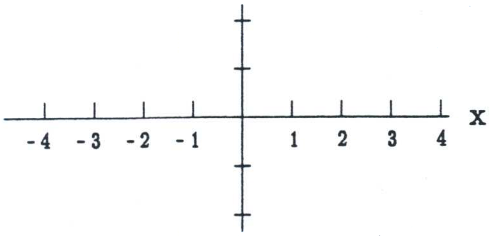
3 $y = .5(x+1)(x-2)(x-4)$

4. $y = (x-2)(x-3)(x-5)(x-6)(x-9)$

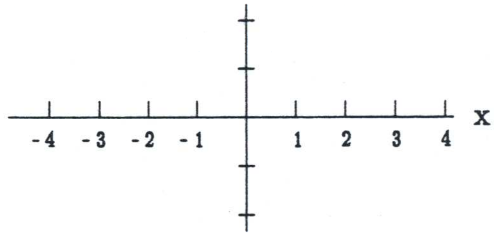
II. Graphing Polynomials having zeros with MULTIPLICITY

Study the graph of $y = x^2$, and $y = x^3$, at $x = 0$.

$y = x^2$



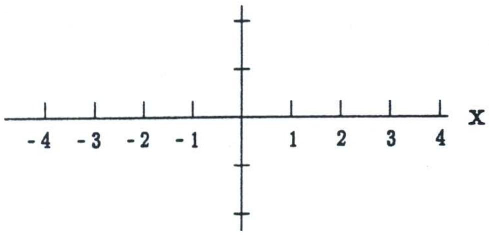
$y = x^3$



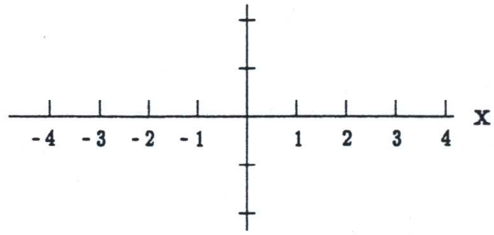
In $y = x^2$, the graph "**bounces back**" , but in $y = x^3$, the graph "**cuts thru**".

What happens in the graphs of $y = x^4$, and $y = x^5$? When does it "bounce back"? When does it "cut thru"?

$y = x^4$



$y = x^5$



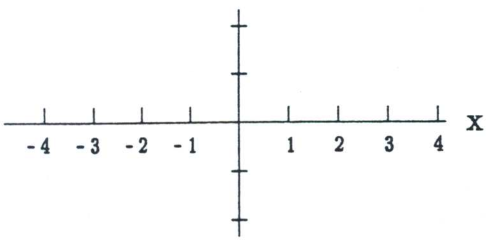
Comment: When a zero is of mulitiplicity 2, or even multiplicity, the graph bounces back. When the zero has odd multiplicity, the graph cuts thru.

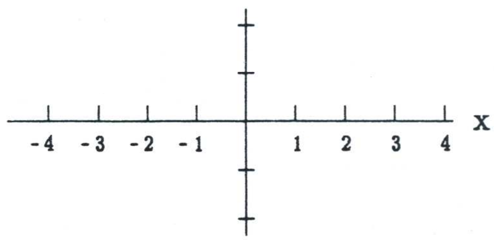
Example 1. $P(x) = (x-1)\,(x-2)^2$.

It is like $y = x^3$
Has zeros at 1, and 2.

Example 2. $(x+1)^2\,(x-2)^3$

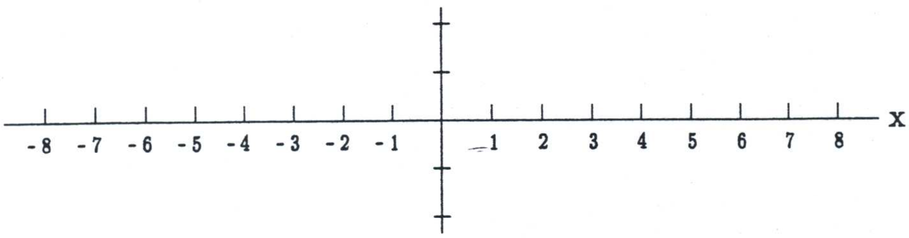
It is like $y = x^5$
Has zeros at -1, and 2.





Example 3. $Q(x) = (x+4)^2 (x+2)(x-2)^3 (x-5)$.

For right and left behaviour, add powers of x. It behaves like $y = x^7$. (We got x^7, by adding the powers of x.) The polynomial has zeros at −4, −2, 2, and 5.
[**Remember**: When the zero is of multiplicity 2, or 4, or 6 (even number) the graph bounces back. When the zero is of multiplicity 1, or 3, or 5, the graph cuts thru.]
In this example the graph will bounce back at x = ______, and cut thru at x =_____.



[use graphing calculator for #3, 4]

Example 4. $y = (x-2)^2 (x-3)^3 (x-5)^4 (x-6)$

(a) For right and left behaviour, add powers of x. It behaves like $y = x^{10}$

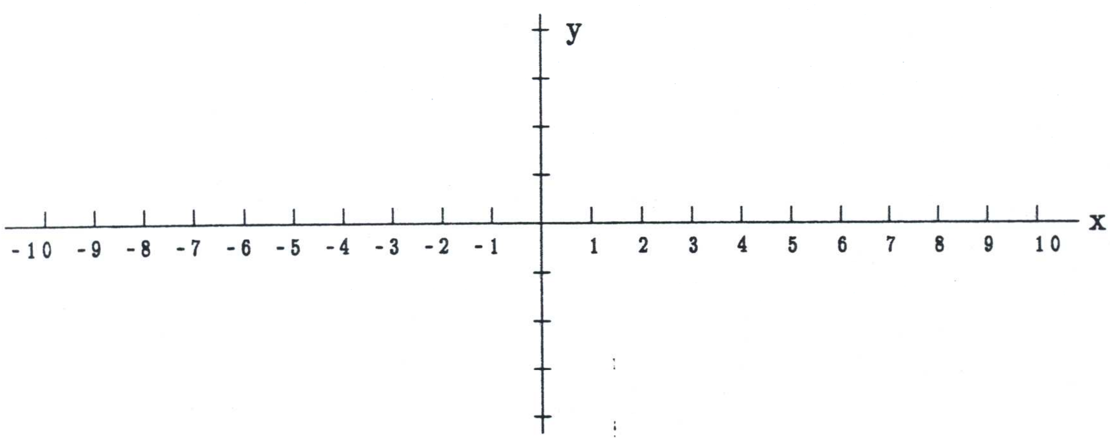
(b) The graph has zeros (meets x axis) at 2,3,5 and 6.

At x=6, the graph ------ (Does it bounce back / cut thru?)

At x=5, the graph ------

At x=3, the graph ------

At x=2, the graph ------



Example 5. $y = 7\,(x-2)\,(x-3)^3\,(x-5)^2\,(x-6)^4$

(a) For right and left behaviour, add powers of x. It behaves like $y = x^{10}$.

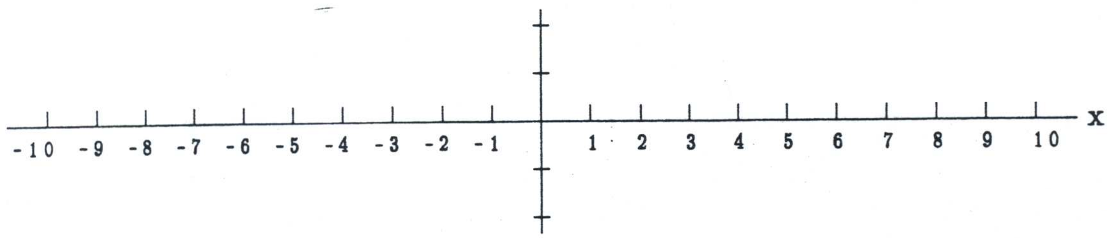
(b) The graph meets x axis at 2,3,5 and 6.

At x=6, the graph ‒ ‒ ‒ ‒ ‒

At x=5, the graph ‒ ‒ ‒ ‒ ‒ ‒

At x=3, the graph ‒ ‒ ‒ ‒ ‒ ‒

At x=2, the graph ‒ ‒ ‒ ‒ ‒ ‒



Example 6. $y = f(x) = (x+4)\,(x+1)^2\,(x-3)^2\,(x-5)^3$

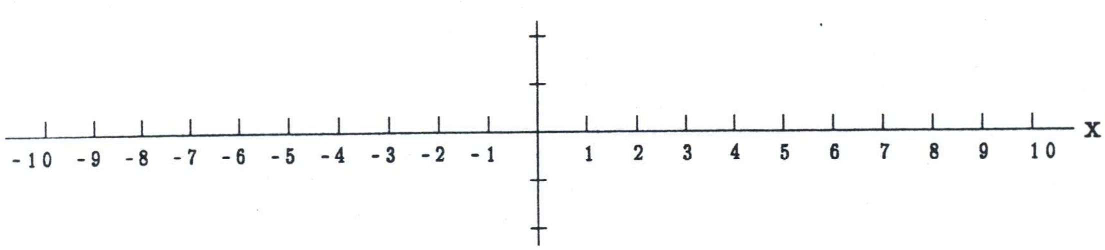
(a) For right and left behaviour, add powers of x. It behaves like $y = x^8$

(b) The graph meets x axis at –4, –1, 3, and 5.

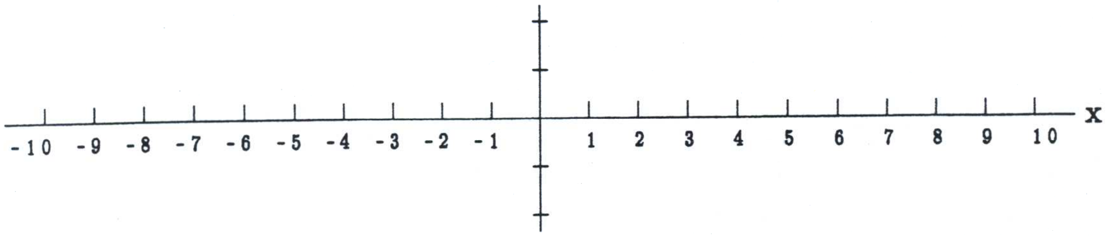
At x = 5, the graph ‒ ‒ ‒ ‒ ‒

At x = 3, the graph ‒ ‒ ‒ ‒ ‒ ‒

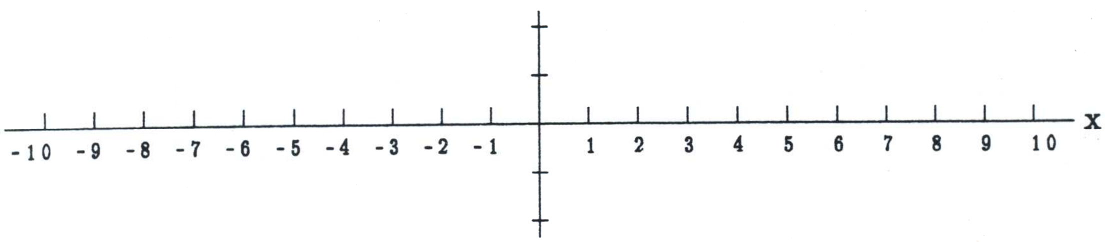
At x = –1, the graph ‒ ‒ ‒ ‒ ‒ ‒



Practice Problem 1. $y = -2\,(x+3)^2\,(x)^3\,(x-2)^4\,(x-6)^5$



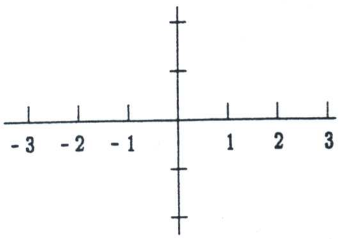
Practice Problem 2: $y = -.5\,(x+5)\,(x+3)^2\,(x-1)^3\,(x-4)^2$

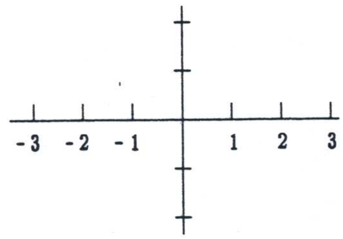


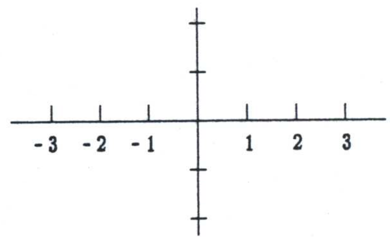
III. Graphs of functions $y = \frac{1}{P(x)}$, where P(x) is a Polynomial

Sketch:

(a) $y = \frac{1}{x}$ (b) $y = \frac{1}{x^2}$ (c) $y = \frac{1}{x^3}$







Note the odd and even functions. Notice what happens at x = 0. When does the graph **flip** on the other side, and when does it **not flip**? In general,

if n is odd, the graph of $y = \frac{1}{x^n}$ is (like $y = \frac{1}{x}$, has flip at x = 0).

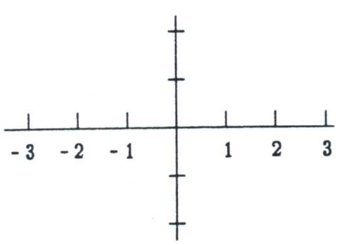
if n is even, the graph is (like $y = \frac{1}{x^2}$, and no flip at 0).

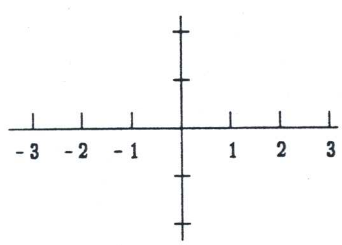
(d) $y = \frac{1}{x^{23}}$

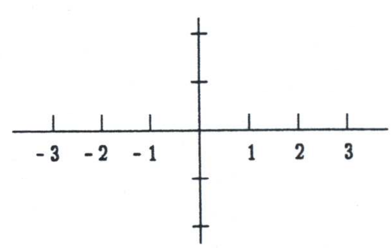
(e) $y = \frac{1}{x^{24}}$

Sketch the following graphs and write comments on the right side.

$y = \frac{1}{x-2}$ $y = \frac{1}{(x-2)^2}$ $y = \frac{1}{(x-2)^3}$







Like $y = \frac{1}{x}$, flip at 2

At x =2 flip

Like $y = \frac{1}{x^2}$

No flip

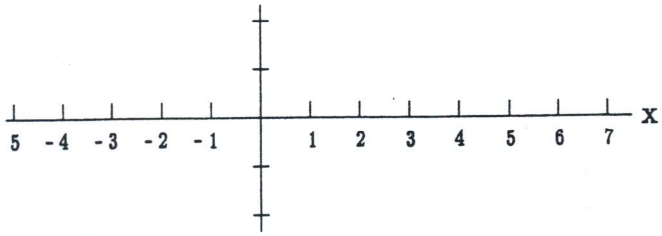
like $y = \frac{1}{x^3}$

flip

What will be graph of $y = \frac{1}{x+3}$?

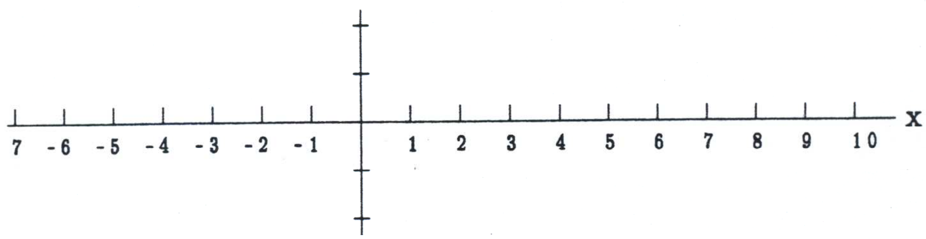
Plot the graph of (a) $y = \dfrac{1}{(x-2)\ (x-4)}$

(like $y = \dfrac{1}{x^2}$, flip at $x = 2$ and $x = 4$)



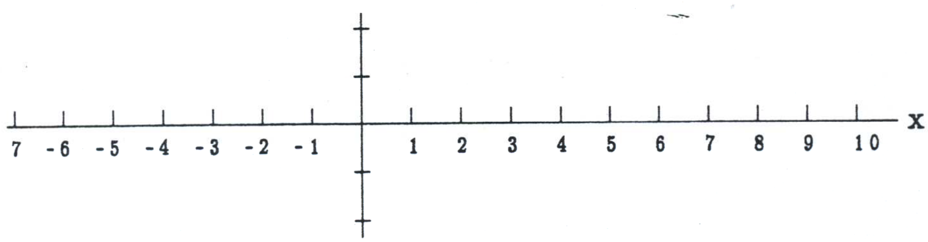
[use graphing calculator]

(b) $y = \dfrac{1}{(x-1)\ (x-3)(\ x-4)\ (x-7)}$ (Like $y = \dfrac{1}{x^4}$. Flip at 1,3,4,7)

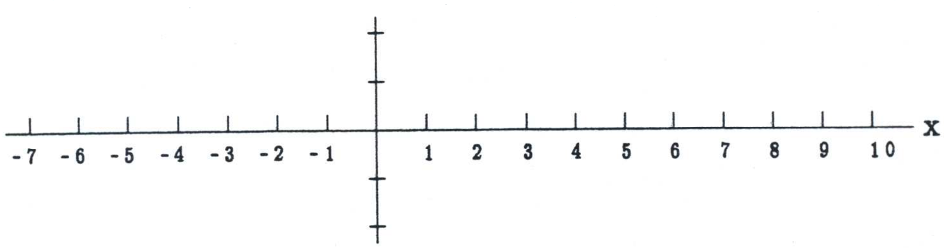


(c) $y = \dfrac{1}{(x-2)\ (x-3)^2\ (x-5)}$

(Like $y = \dfrac{1}{x^4}$. Flip at 2, 5, but not at $x = 3$. This is because as x crosses 3, say if we plug in $x = 3.1$, and then 2.9, the sign of $(x-3)^2$ term does not change.

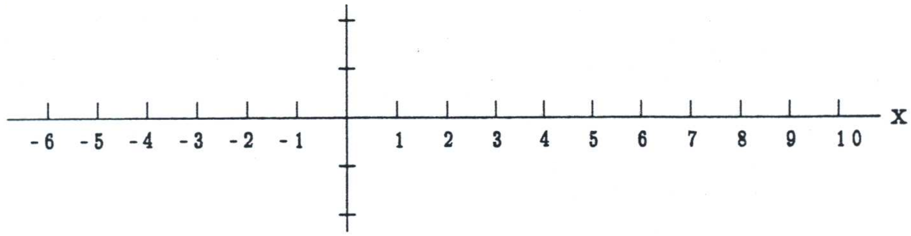


(d) $y = \dfrac{1}{(x+1)(x-2)^3\ (x-5)}$ (Like $y = \dfrac{1}{x^5}$. Flip at ———————)



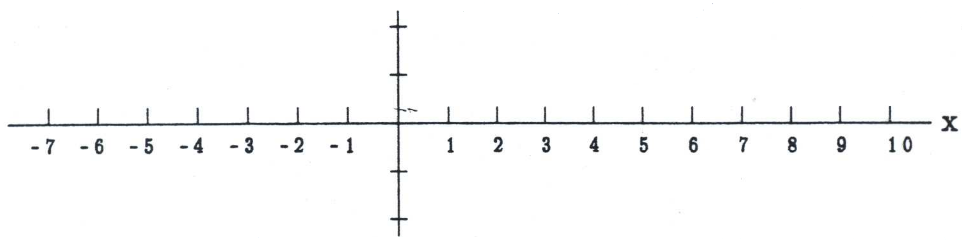
Rule for asymptotes: For **odd** power of the factor, the graph **flips**.
For **even** power of the factor, the graph does **not flip**.

Practice Problem 1. $y = \dfrac{1}{(x + 2)^2\,(x - 3)\,(x - 5)^3}$

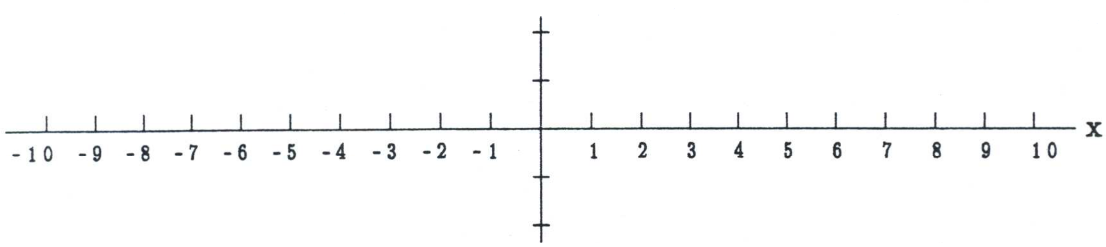


Practice problem 2. $y = \dfrac{1}{(x + 2)\,(x - 3)^2\,(x - 4)^3}$

Practice Problem 3 $y = \dfrac{1}{(x + 2)^2(x - 3)\,(x - 4)^2}$



Practice Problem 4 $y = \dfrac{1}{(x + 6)^2(x + 2)\,(x - 4)^2\,(x-8)^3}$



PART C: RATIONAL FUNCTIONS

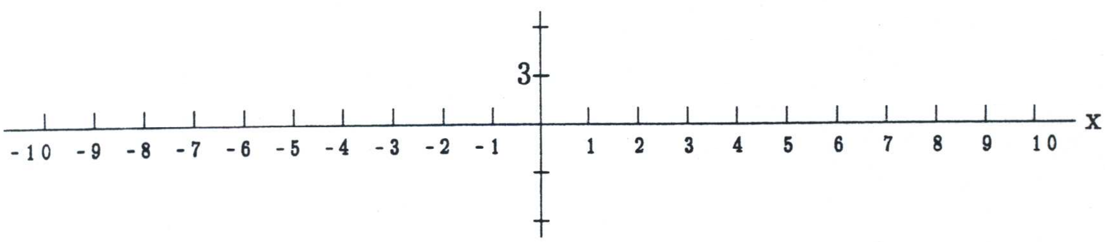
We combine the ideas of:
(i) right and left behaviour
(ii) zeros,
(iii) asymptotes.

Example 1. $y = \dfrac{x^2(x+3)(x-4)(x+9)}{(x-5)(x+6)}$

(i) Right and left behaviour: (Behaves like $y = x^3$, Hint: add up the powers of x)

(ii) has zeros at 0, –3, 4, –9 (bounce back at 0, cut thru at –3, 4, –9).

(iii) has asymptotes at x = 5, x = –6 (flip at both)



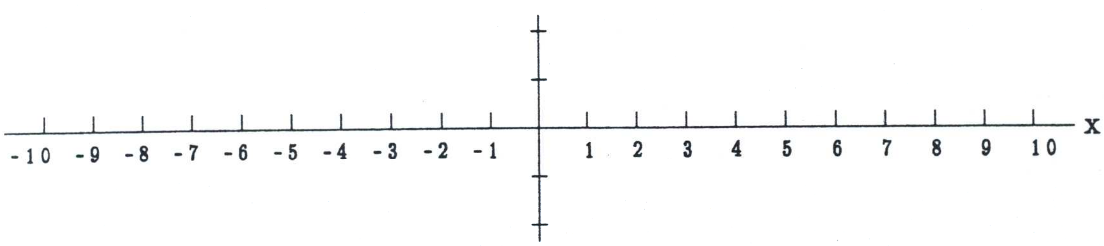
[use graphing calculator]

Example 2. $y = \dfrac{(x+1)\ (x-2)}{(x-4)(x-6)(x-8)(x+2)}$

(i) behaves like $y = \dfrac{1}{x^2} = x^{-2}$

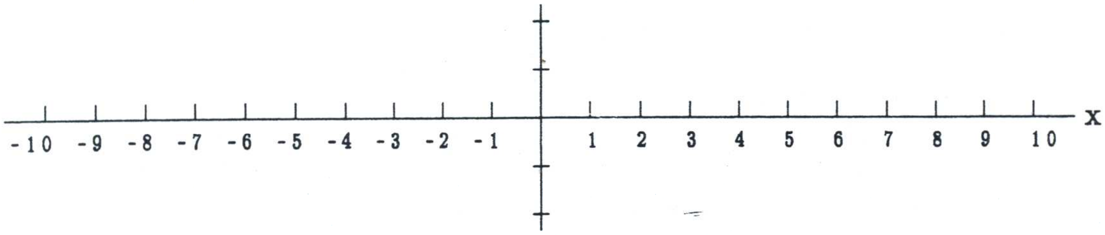
(ii) has zeros at –1, 2, (cuts thru)

(iii) has asymptotes at –2, 4,6,8 (flip at each)

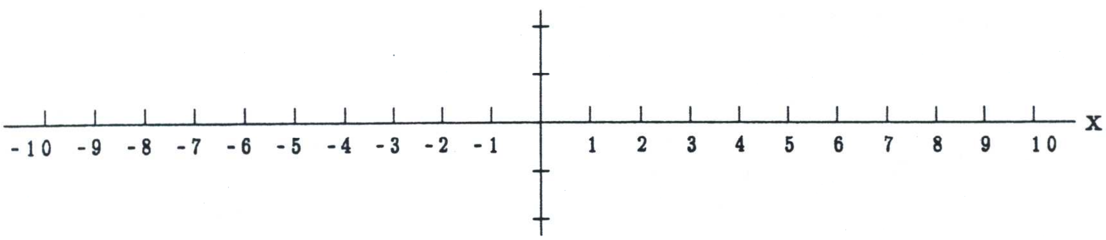


Example 3 $$y = \frac{(x+1)^3\,(x-2)\,(x-4)^2}{(x-1)\quad(x-3)^2}$$

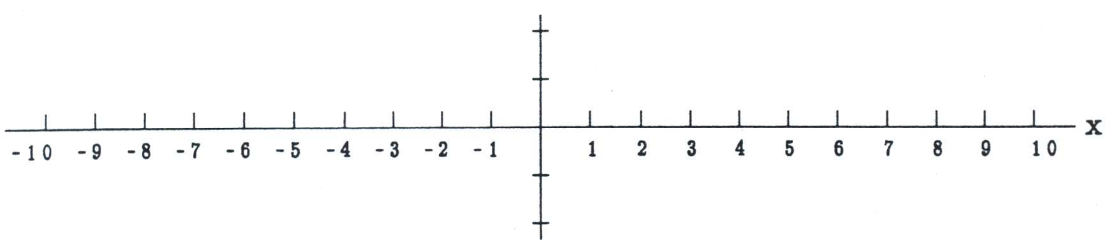
(i) behaves like $y = x^3$
(ii) zeros at −1 (cut thru), 2 (cut thru), 4(bounce back)
(iii) asymptotes at x = 1 (flips) and x = 3 (no flip)



Example 4. $$y = \frac{(x+3)^2\,(x-3)\,(x-5)^3}{(x-1)^2\,(x-2)^2}$$



Example 5 $$y = \frac{(x+1)^2\,(x-2)^2(x-4)}{(x-1)\,(x-3)^2}$$



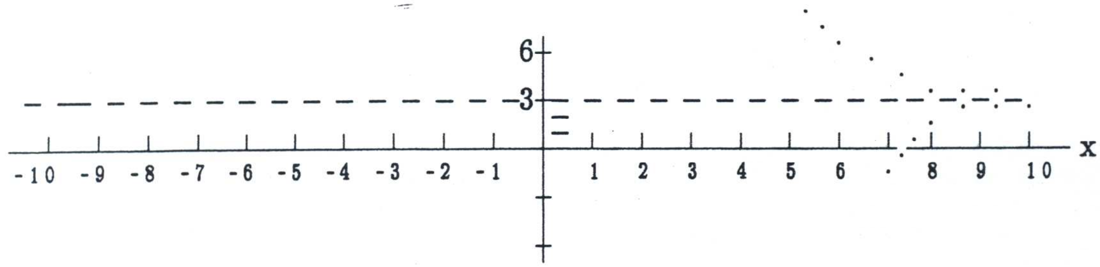
Practice Problem $$y = \frac{(x+1)^4\,(x-2)^2(x-4)^2}{(x-1)^2(x-3)^2\,(x-5)^3}$$

Horizontal asymptotes

Example 1. Consider $y = \frac{3(x-2)^2}{(x-1)(x-5)}$. Right and Left behaviour is like y = 3, i.e. Horizontal Asymptote.

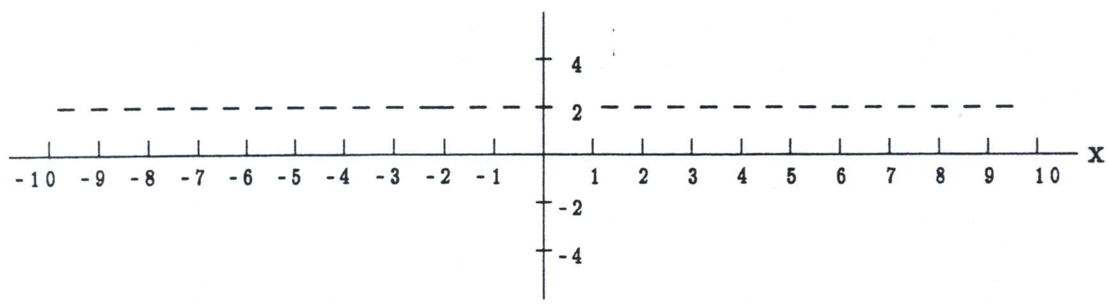
QUESTION: On the right side, does the graph start from below the line or above the line? How is it on the left side? How do we decide?

ANS: To know the graph on the right side, plug a value of x which is on the right side, away from the zeros, and away from the asymptotes. Compute the value of y. If we put x = 10, $y = \frac{3\ (64)}{45}$, this is more than 3. Hence the graph begins above the line y = 3. To find about the left side, plug in a value of x, and decide whether to start the graph below or above the line y = 3



[use graphing calculator]

Example 2: $y = \frac{(x-2)(2x-5)(x-8)}{(x-6)(x-7)(x-10)}$

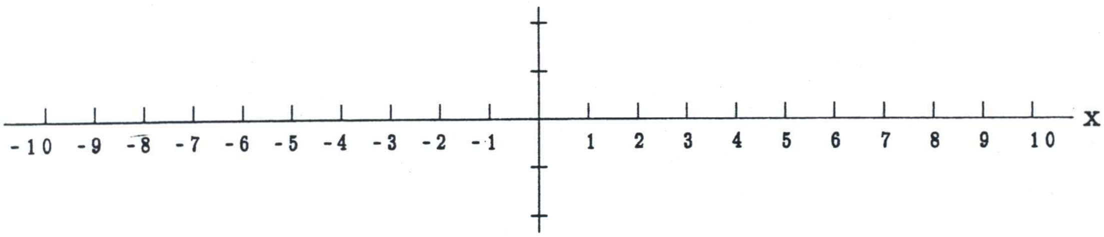


Note that the Numerator is $2x^3$ + lower powers of x, and the Denominator is x^3 + lower powers of x. Hence when x is very large, the curve behaves like y = 2. On the right side, do we start from above the line y = 2 or below the line y = 2? Plug in a value of x, say x = 12, (it is away from all the zeros and the asymptotes). We find that $y = \frac{10\ (19)(4)}{6\ (5)\ (2)} = \frac{76}{6}$, which is more than 2. So we start from above the line y = 2.

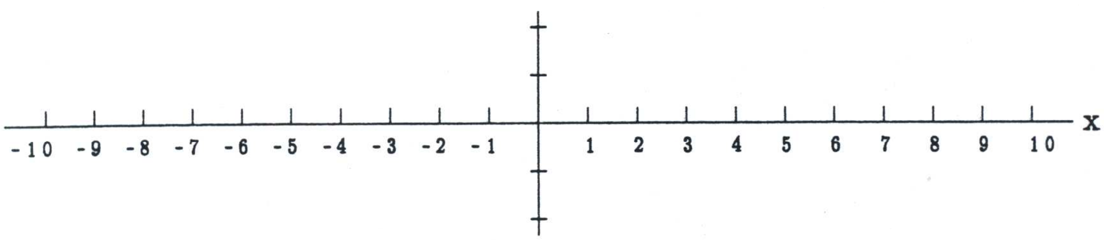
Presence of terms like $(x^2 + 1)$

NOTE: The quantity $(x^2 + 1)$ has highest power 2. But $x^2 + 1$ is never zero. What is its effect in the Numerator, in the Denominator?

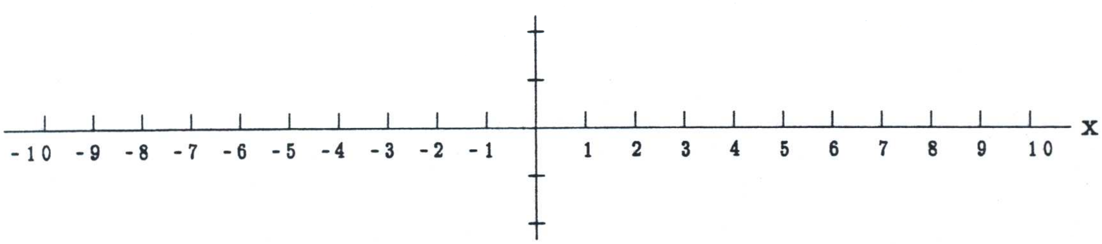
Example 1: $y = \frac{(x-5)\,(x-9)^2}{(x-6)}$ behaves like $y = x^2$. So, $y = \frac{(x-5)\,(x-9)^2(x^2+1)}{(x-6)}$ behaves like $y = x^4$. However, the term $(x^2 + 1)$ does not bring in any extra zeros (since it is never zero).



Example 2. $y = \frac{(x+2)\,(x-3)}{(x-5)^3}$ behaves like $y = \frac{1}{x}$. So $y = \frac{(x+2)\,(x-3)}{(x-5)^3\,(x^2+1)}$ behaves like $y = \frac{1}{x^3}$. However the term $(x^2 + 1)$ does not bring in any extra asymptotes.



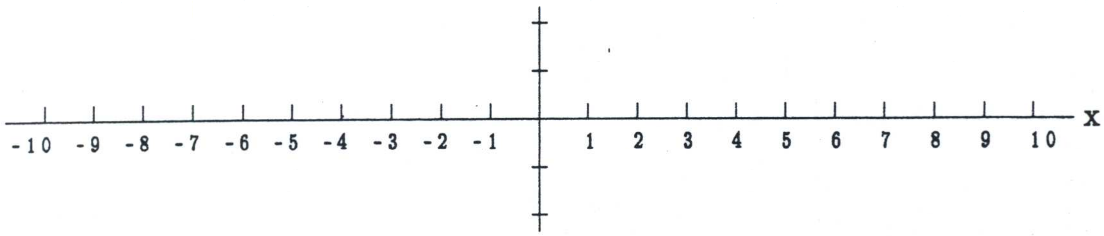
Example 3. $y = \frac{(x-2)^3(x-5)\,(x-9)^2}{(x-3)\,(x-6)(x^2+1)}$ The graph behaves like $y = x^2$. Zeros are at 2, 5, and 9. Asymptotes are at 3, and 6 only. The factor (x^2+1) does not bring in extra asymptotes.



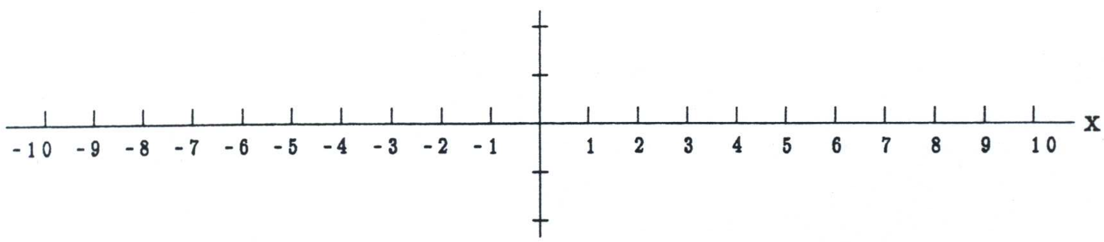
The expression $(x^2 + 1)$ is always positive. Similar expressions are $x^2 + 4$, $x^2 + 4x + 4$, $x^2 + 4x + 9$. In fact any square + something postive will always be positive.

Practice Problems:

(a) $y = \dfrac{(x-4)(x^2+1)}{(x-2)}$



(b) $y = \dfrac{(x-4)\ (x-6)\ (x-8)}{(x-1)\ (x^2+4)}$



(c) $y = \dfrac{(x+4)\ (x-5)\ (x-7)^2\ (x+1)^2}{(x-1)\ (x^2+4)(x^2+6x+10)}$